Math 10B with Professor Stankova
Worksheet, Midterm #2; Wednesday, 4/3/2019
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1 Concepts

1. **Independence**: What does it mean for two events to be independent?

   - What is the difference between independence and mutually disjoint/exclusive? Can two events be both? Neither?
   - What does it mean for three or more events to be independent?

   What does it mean for two random variables to be independent?

   - How do we show two random variables are independent? Dependent?
   - How can we use covariance to show that two random variables are independent? Dependent?

2. **Discrete Random Variables**: Can you draw the picture relating a random variable to a probability space and the PMF?

   - How are the PMF of a random variable and probability function P of a probability space related?
   - How do you calculate and draw a PMF?

   What is the definition of the expected value, variance, standard error, and covariance?

   - How would you explain the expected value, variance, standard error, and covariance in words to a 5 year old?
   - What are some properties of the expected value?
     - When can we split up $E[X + Y] = E[X] + E[Y]$? When can we split up $E[XY] = E[X]E[Y]$?
   - What is the "shortcut" formula for the variance? Can you prove it?
   - When can we split up $Var[X + Y] = Var[X] + Var[Y]$? When can we split up $Var[XY] = Var[X]Var[Y]$?
   - What is the expected value and variance of a constant?
   - What is the relationship between the variance and standard error? Why do we have both of them?
   - How can we recover the variance from the covariance?

Fill out the following table:

Distribution	Range(X)	PMF	$E(X)$	Variance	$SE(X)$	Example
Uniform						
Bernoulli Trial						
Binomial						
Geometric						
Hyper-Geometric						
Poisson						

3. **Limit Theorems**: What does i.i.d stand for?

- What does $\bar{X}$ represent?
- What do $\bar{\mu}, \bar{\sigma}$ represent?
- How do $\mu, \sigma, \bar{\mu}, \bar{\sigma}, X, \bar{X}$ relate to each other?
- What is the difference between $\mu, \bar{\mu}, \bar{X}$?
- Can we say anything about whether σ or $\bar{\sigma}$ is bigger?

What do the Central Limit Theorem (CLT) and Law of Large Numbers (LoLN) state?

- Can we use CLT to prove LoLN or vice versa?
- How does the CLT relate to z-scores? Is this relationship exact?
- Pictorially, what are the CLT and LoLN saying?

4. **Continuous Random Variables:** How do the PMF and PDF differ?

- What properties must a function satisfy to be a PDF? A CDF?
- What is the relationship between a PDF and CDF?
- How do we calculate the median and mean?
- How do the median and mean relate? When are they equal? When is the mean bigger? When is the mean smaller?
- How are histograms related to PDFs?
- Find the PDF, CDF, mean, and median of the following distributions:
  - Continuous uniform
  - Dart-player
  - Normal distribution
  - Exponential
  - Pareto

2 Problems

2.1 Discrete Distributions

5. Suppose that we roll two die and let X be equal to the maximum of the two rolls. Find $P(X \in \{1, 3, 5\})$ and draw the PMF for X. What is the expected value and standard error of X?

6. I draw 5 cards from a deck of cards. Let X be the number of hearts I draw. What is the range of X and draw the PMF of X. Use this to find the probability that I draw at least 2 hearts.

7. Suppose I have a weighted 4 sided die that lands on 1 with probability $\frac{1}{2}$ and lands on $2, 3, 4$ with equal probability. Let X be the value of the die when I roll it once. What is the PMF for X. What is $E[X]$ and $Var(X)$?

8. I am picking cards out of a deck. What is the probability that I pull out 2 kings out of 8 cards if I pull with replacement? What about without replacement?

9. What is the probability that first king is the third card I draw (with replacement)?

10. In a class of 40 males and 60 females, I give out 4 awards randomly. What is the probability that females will win all 4 awards if the awards must go to different people? What about if the same person can win multiple awards?

11. I am playing the lottery and I have a 0.1% chance of winning. What is the probability that I need to play at least 100 times until winning?

12. In a class of 50 males and 90 females, I give out 4 awards randomly. What is the probability that females will win 2 awards if the awards must go to different people? What about if the same person can win multiple awards?

13. At Berkeley, 3/4 of the population is undergraduates. I cold call someone at random and ask for their age. What is the probability that I have to call 10 people before I call an undergraduate? What is the probability that I call 4 undergraduates out of 10 people I call (if I can call someone more than once)?

14. Suppose that the probability that a child is a boy is 0.51. What is the probability that a family of five children has at least one girl? What is the probability that the family of 5 has all children of the same sex?

15. In a dorm of 100 people, there are 20 people who are underage. I go to a party with 40 people. What is the probability that there is at least one underage person there?

16. In a class of 30 students, I split them up into 6 groups of 5. What is the expected number of days of splitting them up randomly into new groups of 5 before I split them up into the same groups again (assume that the groups are indistinguishable)?

17. When a cell undergoes mitosis, the number of mutations that occurs is Poisson distributed and an average of 8 mutations occur. What is the probability that no more than 1 mutation occurs when two cells divide?

18. I roll a fair 6-sided die over and over again until I roll a 6. What is the probability that it takes me more than 10 tries? What is the expected number of total rolls I need and what is the variance?

19. I am throwing darts some number of times and suppose that I expect to hit the dart board 20 times with a standard error of 2 times. What is the probability that I hit the dart board on a single throw?

20. When creating this worksheet, the probability a particular problem has a typo is 1%. What is the probability that I had at most one typo if I created 100 problems?

21. The number of chocolate chips in a cookie is Poisson distributed with an average of 15 chocolate chips. What is the probability that you pick up a cookie with only 10 chocolate chips in it?

22. The number of errors on a page is Poisson distributed with approximately 1 error per 50 pages of a book. What is the probability that a novel of 300 pages contains at most 1 error?

23. This weekend I went to the marina and saw 20 boats at sea out of the 200 total docked at the marina. The next weekend, I go back and see 30 at sea this time. What is the probability that 6 of those boats were also at sea the previous weekend?

24. Suppose that I am trying to make a half court shot and I have a 1% chance of making it. I try 800 times total and assume the trials are independent (I don't get tired). What is the exact probability that I make exactly 6 shots? Using the Poisson distribution, approximate the probability that I make 6 shots.

25. I roll two fair 6 sided die. What is the expected value of their product?

26. In a class of 30 students, I split them up into 3 groups of 10 on Tuesday. Today, Thursday, I split them up into 6 groups of 5 randomly. What is the expected number of people in your new group were in your old group on Tuesday?

27. While pulling out of a box of cookies, what is the expected number of cookies I have to pull out before I pull out an oatmeal raisin if 25% of cookies are oatmeal raisin and I choose with replacement? What is the variance?

28. What is the expected number of aces I have when I draw 5 cards out of a deck?

29. Suppose that X is a binomial random variable with expected value 20 and variance 4. What is $P(X = 3)$?

30. In a safari, safari-keepers have caught and tagged 300 rhinos. On a safari, out of the 15 different rhinos you see, there are 5 of them expected to be tagged. How many rhinos are there at the safari?

31. Suppose that I flip a fair coin 8 times. Let T be the number of tails I get and H the number of heads. Calculate $E[T], E[H], Var[T], Var[H], Var[T+H]$. Now calculate $E[T-H]$ and $Var[T-H]$.

32. Prove the short cut formula for variance from the definition of variance.

2.2 Limit Theorems

33. Let $X_1, \ldots, X_4$ be i.i.d Bernoulli trials with $p = \frac{3}{4}$. Let $\bar{X}$ be the average of them. What is $Var[\bar{X}]$? Find $Cov(X_1, \bar{X})$ (Hint: Write $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$).

34. Let f be normally distributed with mean 3 and standard error 5. Calculate the probability $P(X \geq 0)$.

35. Let f be normally distributed with mean 2 and standard error 1. Calculate the probability $P(X \leq 0)$.

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38. Let $X_1, \ldots, X_4$ be i.i.d Bernoulli trials with $p = \frac{3}{4}$. Let $\bar{X}$ be the average of them. What is $Var[\bar{X}]$? Find $Cov(X_1, \bar{X})$ (Hint: Write $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$).

39. Suppose the weight of newborns is distributed with an average weight of 8 ounces and a standard deviation of 1 ounce. Today, there were 25 babies born at the Berkeley hospital. What is the probability that the average weight of these newborns is less than 7.5 ounces?

40. Suppose that the average lifespan of a human is 75 years with a standard deviation of 10 years. What is the probability that in a class of 25 students, they will on average live longer than 80 years?

41. The newest Berkeley quarterback throws an average of 0.9 TDs/game with a standard deviation of 1. What is the probability that he averages at least 1 TD/game next season (16 total games)?

42. Suppose that the average shopper spends 100 dollars during Black Friday, with a standard deviation of 50 dollars. What is the probability that a random sample of 25 shoppers will have spent more than $3000?

43. Suppose that on the most recent midterm, the average was 60 and the standard deviation 20. What is the probability that a class of 25 had an average score of at least 66?

2.3 Continuous Random Variables

44. Let $f(x) = \begin{cases} ce^{-x} & -1 \leq x \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

45. Let $c(x) = \begin{cases} \frac{c}{x^4} & x \leq -1 \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

46. Let $f(x) = \frac{c}{1+x^2}$ for $x \geq 0$ and 0 otherwise. Find c such that $f(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

3 True/False

47. True False To partition a set Ω into a disjoint union of subsets $B_1, B_2, \ldots, B_n$, means that the intersection of these sets is empty; i.e., $B_1 \cap B_2 \cap \cdots \cap B_n = \emptyset$.

48. True False Two disjoint events could be independent, but two independent events can never be disjoint.

49. True False If a fair coin comes up Heads six times in a row, it is more likely that it will come up Tails than Heads on the 7th flip.

50. True False Contrary to how we may use the word "dependent" in everyday life; e.g., event A could be dependent on event B, yet event B may not be dependent on event A; in math "dependent" is a symmetric relation; i.e., A is dependent with B if and only B is dependent with A.

51. True False If A and B are independent events, their complements are also bound to be independent, and to prove this we need a general argument since an example is not sufficient here.

52. True False If A and B are independent events, $\overline{A}$ and B may fail to be independent, but to prove this we need just one counterexample, not a general proof.

53. True False If any pair of events among $A_1, A_2, ..., A_n$ are independent, then all events are independent.

54. True False A random variable (RV) on a probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$ that satisfies certain rules and is related to the probability function P.

55. True False A RV X could be the only source of data for an outcome space Ω and hence could be very useful in understanding better X 's domain.

56. True False The notation "$X \in E$" means that the RV X starts from event $E \subseteq \Omega$ and lands in $\mathbb{R}$.

57. True False The notation "$X^{-1}(E)$" for a RV X and event $E \subseteq \Omega$ means to take set $B \subseteq \Omega$ of the reciprocals of all elements in E that are in the range of X .

58. True False The PMF of a RV X on probability space (Ω, P) is a third function $f : \mathbb{R} \to [0,1]$ such that the composition of X followed by f on any $\omega \in \Omega$ is equal to P; i.e., such that $f(X(\omega)) = P(\omega)$.

59. True False It is possible that $f(x) > P(x)$ for some $\omega \in \Omega$ and the corresponding $x = X(\omega) \in \mathbb{R}$ where X a discrete RV on (Ω, P) with PMF f .

60. True False To show that two RV's $X, Y : \Omega \to \mathbb{R}$ are independent on (Ω, P) , we can find two subsets $E, F \subseteq \mathbb{R}$ for which $P(X \in E \text{ and } Y \in F) = P(X \in E) \cdot P(Y \in F)$.

61. True False Two Bernoulli trials are independent only if the probability of success and failure are each $\frac{1}{2}$.

62. True False The product X and the sum Y of the values of two flips of a fair coin (H=1, T=0) are dependent random variables.

63. True False To turn the experiment of "rolling a die once" into a Bernoulli trial, we need to split its outcome space into two disjoint subsets and declare one of them a success.

64. True False Several Bernoulli trials performed on one element at a time from a large outcome space Ω, without replacement, are approximately independent because what happens in one Bernoulli trial hardly affects the ratio of "successes" to "failures" in the remainder of the population.

65. True False The probability of having 20 women within randomly selected 40 people is about 50%, assuming that there is an equal number of women and men on Earth.

66. True False The hypergeometric distribution describes the probability of k "successes" in n random draws without replacement from a population of size N that contains exactly m "successful" objects.

67. True False While the hypergeometric and binomial probabilities depend each on 3 parameters and 1 (input) variable, the Poisson probability depends only on 1 parameter and 1 (input) variable.

68. True False To approximate well the probability of k successes in a large number n of independent Bernoulli trials, each with individual probability of success p that is relatively small, we can use the formula $\frac{(np)^k e^{-np}}{n!}$.

69. True False $E(X-Y) = E(X) - E(Y)$ for any R.V.s X and Y, regardless of whether they are independent or not.

70. True False $Var(X-Y) = Var(X) - Var(Y)$ for any independent R.V.s X and Y.

71. True False $Var(X) = E(X^2) - E^2(X)$ holds true because, essentially, the expected value has linearity properties.

72. True False Splitting a R.V. X as a sum of simpler R.V.'s X_i could be advantageous when we want to compute $Var(X)$, but we need to be careful that these X_i's are independent.

73. True False If X is the geometric R.V., then $Var(X) = \frac{1-p}{p^2}$.

74. True False If X is the hypergeometric R.V. in variable k and with parameters m, n, N, then $N = \frac{mn}{E(X)}$.

75. True False If Santa Claus randomly throws n presents into n chimneys (one present per chimney), on the average one home will receive their intended present, regardless of how large or small n is.

76. True False For any independent R.V's X and Y, we have $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$, but $Var(XY) = Var(X) \cdot Var(Y)$ only if $E(X) = 0 = E(Y)$ or $X = 0$ or $Y = 0$ or both X and Y are constants.

77. True False As long as several RV's $X_1, X_2, ..., X_n$ are identically distributed, their average RV $\overline{X}$ will have the same mean as each of them, but the standard error of $\overline{X}$ may or may not be equal to $\frac{SE(X_3)}{\sqrt{n}}$.

78. True False The formula for the mean of the average $\overline{X} = \frac{X_1+X_2+\cdots+X_n}{n}$ of independent identically distributed RV's has $\sqrt{n}$ in the denominator.

79. True False The formula for the variance $Var(X + Y) = Var(X) + Var(Y)$ works regardless of whether the RV's X and Y are independent or not.

80. True False To approximate the height of a tall tree (using similar triangles and measurements along the ground – without climbing the tree!) it is better to ask several people to do it independently of each other and then to average their results, than to do it once just by yourself.

81. True False If $\overline{X}$ is the average of n IIDRVs, each with mean μ and standard error σ, then for n large the normalized distribution $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ has a very small standard error.

82. True False According to the Central Limit Theorem, the normalized distribution $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is the standard normal distribution for n large, where $\overline{X}$ is the average of n independent, identically distributed variables, each with mean μ and standard error $\sigma/\sqrt{n}$.

83. True False As n grows the average $\overline{X}$ of IIDRV's $X_1, X_2, X_3..., X_n$ becomes more wide-spread, since we are taking larger and larger samples and incorporating more data into our calculations.

84. True False The Central Limit Theorem states that for large n, the normalized distribution $Z = \frac{\bar{X}-\mu}{\bar{\sigma}}$ is the standard normal distribution.

85. True False Histograms are always defined over intervals of the form $[a, b]$ with $a \geq 0$ because probabilities are always non-negative.

86. True False The height of a rectangle in a histogram equals $\frac{\text{the amount of the data corresponding to the subinterval}}{\text{the width of the subinterval}}$, because all rectangular areas in a histogram must sum up to 1.

87. True False $P(x) = e^{-x^2}$ is a PDF on $\mathbb{R}$.

88. True False We allow for a PDF to occasionally "peak" above 1 (e.g., $f(x) > 1$ for some $x \in \mathbb{R}$), but a PMF is forbidden to do that!

89. True False A PDF $f(x)$ can have values above 1, but this can happen only at finitely many places $x_1, x_2, \ldots, x_n \in \mathbb{R}$

90. True False Uniform PDFs are defined by $f(x) = c$ for all $x \in \mathbb{R}$.

91. True False Shifting the bell-shaped PDF $f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$ to the left by 5 units results in another PDF $g(x) = \frac{1}{\sqrt{\pi}}e^{-(x+5)^2}$ centered at $x = 5$.

92. True False For a PDF to be centered at $x = a$, it means that a is the median of the CDF.

93. True False CDFs behave in general like antiderivaties of their PDFs, but there are some situations where the CDF is not continuous and hence there is no way it can be an antiderivative of a PDF.

94. True False The formula $P(a \leq X \leq b) = F(b) - F(a)$ for a CDF $F(x)$ works because $F(x)$ can be essentially thoughout of as the "area-so-far" function for a PDF.

95. True False To prove that a function $F(x)$ on $\mathbb{R}$ is a CDF, we need only to confirm that it is non-decreasing on $\mathbb{R}$, attains only non-negative values, and that its value tends to 1 and 0 as $x \to \infty$ and $x \to -\infty$ correspondingly.

96. True False The CDF of the bell-shaped PDF $f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$ has a graph that is increasing and looks like a solution to the classic logistic model with carrying capacity $K = 1$, but it cannot be exactly equal to it because there is no elementary function of an antiderivative of e^{-x^2} yet we have derived the formula $g(x) = \frac{1}{1+Ae^{-kt}}$ for these logistic model solutions.

97. True False There are at least three ways to compute the probability of a person to be born in May: using a discrete random variable, and using the PDF or the CDF of a continuous random variable.

98. True False The formula for the mean of a continuous random variable is a limit version of the mean for a discrete random variable; but while the latter always exists for a finite amount of data, the former may not exist for certain continuous random variables.

99. True False If the mean is larger than the median, the distribution tends to be more spread away on the right and more clustered together on the left.

100. True False If we make the area between a PDF and the x-axis out of uniform cardboard material and make an infinite seesaw out of the x-axis, the point on the x-axis where the seesaw will balance is the median of the distribution because there is an equal material to the left and to the right of the median.

101. True False Exam distributions of large classes tend to have smaller means than medians when the medians are higher than 50% of the maximum possible score.

102. True False The Pareto distribution fails to have a well-defined mean when the constant $a \geq 2$.

103. True False Improper integrals resurface when we want to compute probabilities of discrete random variables with finitely many values.

104. True False For a symmetric distribution, we do not have to calculate the mean because it will always equal the median.

105. True False $\frac{1}{\sqrt{2\pi}}$ ensures that the formula for the normal distribution indeed represents a valid PDF.

106. True False z scores are not suitable for computing probabilities of the type $P(-\infty \leq X \leq a)$ or $P(b \leq X \leq \mu)$ for arbitrary normal distributions.

107. True False Normal distributions are defined only for positive X; yet, when converted to the standard normal distribution, they may be defined for negative X too.

108. True False PMFs replace PDFs when moving from continuous to discrete random variables.

109. True False CDFs can be defined by the same probability formula for both discrete and continuous variables; however, at the next step when actually computing the CDFs, one must be careful to use correspondingly PMFs with integrals and PDFs with appropriate summations.

110. True False The target spaces for PDFs, PMFs, and probability functions are all the same.

111. True False The domains of PDFs and PMFs are the corresponding outcome spaces Ω.